Quantum Groups Seminar

Growth in tensor powers

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(jt with Kevin Coulembier, Pavel Etingof, Daniel Tubbenhauer)

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$$\frac{1}{M}(\dim(V)^n - Kr^n) \leq b_n(V) \leq \dim(V)^n$$

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$$\mathrm{Ker}\,(V) \subset \mathrm{Ker}\,(V \otimes V^*) \subset \Gamma$$
- there exists $M \in \mathbb{Z}_{>0}$ such that for any $r = 0, 1, \ldots M-1$ the subsequence $\{\frac{b_n(V)}{\dim(V)^n}\}_{n \equiv r \pmod{M}}$ converges

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Generalization (P. Biane (1993) et al):

Γ reductive over $F = \mathbb{C}$: $b_n(V) \sim K(V) \frac{\dim(V)^n}{n^{b/2}}$ where $b = |R_+|$ integer

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Assume char $F = 0$ and there is $K > 0$ such that $b_n(V) \geq K \dim(V)^n$. Then Zariski closure of the image of Γ in $GL(V)$ is a finite group extended by torus. Equivalently, $\Gamma \supset \Gamma_0$ such that $[\Gamma : \Gamma_0] < \infty$ and the image of Γ_0 consists of simultaneously diagonalizable matrices.

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where $\alpha_2 \approx 0.7075$, $\alpha_3 \approx 0.6845$

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H. H. Andersen/S. Donkin: combinatorial description of $b_n(V)$
Numerically: $K' \frac{2^n}{n^{\alpha_p}} \leq b_n(V) \leq K'' \frac{2^n}{n^{\alpha_p}}$ for some α_p and $K', K'' > 0$
where $\alpha_2 \approx 0.7075$, $\alpha_3 \approx 0.6845$

Conjecture (P. Etingof): $K' \frac{2^n}{n^{\alpha_p}} \leq b_n(V) \leq K'' \frac{2^n}{n^{\alpha_p}}$ for some $K', K'' > 0$

$$\boxed{\alpha_2 = \frac{1}{2} \log_2 \frac{8}{3}} \qquad \boxed{\alpha_3 = \frac{1}{2} \log_3 \frac{9}{2}} \qquad \boxed{\alpha_p = \frac{1}{2} \log_p \frac{2p^2}{p+1}}$$

however $b_n(V) \not\sim K \frac{2^n}{n^{\alpha_p}}$

Question: What about other representations of $SL(2)$?
Is the exponent α_p universal?
Question What about other groups? e.g. $SL(3)$?

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Consider $c'_n(V) =$ number of non-projective summands in $V^{\otimes n}$
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• Open True/False question: is $\gamma(V) = \gamma'(V)$ for all V?

Example

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Exercise. Compute $\gamma(J_2)$ and $\gamma(J_4)$ (of course $\gamma(J_1) = 1$ and $\gamma(J_5) = 0$)

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$$c(V) = \liminf_{n\to\infty} \frac{d_n(V)}{\delta(V)^n} > 0$$

Conjecture: $c(V) \geq e^{-a_p\delta(V)}$ for some $a_p \in \mathbb{R}_{>0}$.
This is true for $p = 2$ and $p = 3$ with

$$a_2 = \frac{4\ln(3)}{3} \approx 1.464, \quad a_3 = 24$$

For $p \geq 5$ we have $c(V) \geq \exp(-a_p\delta(V) - \frac{\pi\ln(2)}{2}(p-2)\delta(V)^2)$

More

Theorem (K. Coulembier, P. Etingof, V. O.)

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Corollary: $\delta(V)$ is finitely computable (finitely many $d_n(V)$ are required)

Thanks for listening!